MATHEMATICS

NOTES ON BANACH FUNCTION SPACES, IX

BY

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In the present note some of the results in the preceding Note VIII (these Proceedings 67, p. 104) are extended and generalized to Riesz spaces which are not necessarily Dedekind complete (and in many cases not even σ-Dedekind complete). Although the results in Note VIII would suffice for most of the applications which are to follow (in particular applications to Banach function spaces), we have decided to include these extensions in this series of notes.

29. *Normal subspaces generated by ideals; Archimedean Riesz spaces*

If L is a Dedekind complete Riesz space and A is an ideal in L, then A^{pp} is the smallest normal subspace including A, and $L = A^p \oplus A^{pp}$ (cf. Corollary 17.3 in Note VI). The subspace A^{pp} is also characterized by the fact that the positive elements of A^{pp} are exactly the elements $0 \leqslant u \in L$ satisfying $u = \sup (v : v \in A, v \leqslant u)$. The proof is easy. Since L is Dedekind complete, the element $u' = \sup (v : v \in A, v \leqslant u)$ exists for any given $0 \leqslant u \in A^{pp}$, and evidently we have $u' \leqslant u$, $u' \in A^{pp}$, so $0 \leqslant u - u' \in A^{pp}$. It remains to prove that $u - u' \in A^p$, since then it follows immediately that $u' = u$. We have to prove, therefore, that $\inf(u-u', v) = 0$ for every $0 \leqslant v \in A$. Assume that $w_0 = \inf (u - u', v_0) > 0$ for some $v_0 \in A$. Then $w_0 \in A$, and so for every $v \in A$, $v \leqslant u$, we have $v + w_0 \in A$ and $v + w_0 \leqslant u' + (u - u') = u$. Hence $u' + w_0 = \sup (v + w_0 : v \in A, v \leqslant u) \leqslant u'$. Contradiction.

Now, assume that the Riesz space L is not necessarily Dedekind complete. The smallest normal subspace including any given ideal $A \subset L$ will be denoted by $\{A\}$. It may occur that $\{A\}$ is properly included in A^{pp}. By way of example, let L be the plane ordered lexicographically (i.e., $(x, y) \geqslant 0$ whenever $x > 0$ or $x = 0, y \geqslant 0$). The only ideals in L are $\{0\}$, the y-axis and L itself, and all of these ideals are normal subspaces. If A is the y-axis, then $\{A\} = A$, but $A^p = \{0\}$ and hence $A^{pp} = L$.

It remains true, however, that if A is an ideal in L, then the positive elements in $\{A\}$ are exactly the elements $0 \leqslant u \in L$ satisfying

$$u = \sup (v : v \in A, v \leqslant u),$$

but the above proof is no longer available.

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Lemma 29.1. *Let A be an ideal in the Riesz space L, and $\{A\}$ the normal subspace generated by A. Then any $0 \leqslant u \in \{A\}$ satisfies $u = \sup(v : v \in A, v \leqslant u)$, and conversely, by the definition of a normal subspace, any $0 \leqslant u \in L$ satisfying this condition is an element of $\{A\}$.*

Proof. Let B be the set of all $0 \leqslant u \in L$ satisfying $0 \leqslant v_\tau \uparrow u$ for some directed set $\{v_\tau\} \subset A$, and let $A^\wedge$ be the set of all $u_1 - u_2$ with $u_1, u_2 \in B$. Then $A \subset A^\wedge \subset \{A\}$, and we assert that B consists of all $0 \leqslant u \in L$ satisfying $u = \sup(v : v \in A, v \leqslant u)$. Indeed, if u satisfies this condition, then $u \in B$ since $(v : v \in A, v \leqslant u)$ is directed upwards. Conversely, $u \in B$ implies that $0 \leqslant v_\tau \uparrow u$ for some directed set $\{v_\tau\} \subset A$, and so any upper bound of $(v : v \in A, v \leqslant u)$ is at least u. On the other hand u is an upper bound, and hence $u = \sup(v : v \in A, v \leqslant u)$. Note also that $0 \leqslant u' \leqslant u \in B$ implies $u' \in B$ by the definition of B. But then also $0 \leqslant u - u' \in B$, and this shows that any positive element of $A^\wedge$ is an element of B, i.e., $B = (A^\wedge)^+$. The proof will be complete, therefore, if we show that $A^\wedge$ is a normal subspace, since this will imply that $A^\wedge = \{A\}$, so $B = \{A\}^+$.

In order to show that $A^\wedge$ is a linear subspace, it is sufficient to prove that $u_1, u_2 \in B$ implies $u_1 + u_2 \in B$. Denoting an arbitrary upper bound of $(v : v \in A, v \leqslant u_1 + u_2)$ by p, we have $p \geqslant v_1 + v_2$ for any pair $v_1, v_2 \in A$, $v_1 \leqslant u_1$, $v_2 \leqslant u_2$, so $p \geqslant u_1 + u_2$. On the other hand $u_1 + u_2$ is an upper bound, and hence the least upper bound. This shows that $u_1 + u_2 \in B$.

In order to show that $A^\wedge$ is an ideal, assume that $f \in A^\wedge$, so $f = u_1 - u_2$ with $0 \leqslant u_1, u_2 \in B$. Since $0 \leqslant f^+ \leqslant u_1$ and $0 \leqslant f^- \leqslant u_2$, it follows that $f^+, f^- \in B$, so $|f| \in B$. Hence, if $f \in A^\wedge$ and $|g| \leqslant |f|$, then $g^+, g^- \in B$, and so $g \in A^\wedge$.

Finally, in order to show that $A^\wedge$ is a normal subspace, assume that $0 \leqslant u_\tau \uparrow u$ with $u_\tau \in B$ for all τ. Then

$$
\begin{aligned}
u = \sup u_\tau = \sup\nolimits_\tau \sup(v : v \in A, v \leqslant u_\tau) = \\
= \sup(v : v \in A, v \leqslant u_\tau \text{ for some } \tau).
\end{aligned}
$$

Since the last set is directed upwards, we have $u \in B$. This completes the proof.

In the next lemma we collect, for later reference, some minor results. In all statements L is an arbitrary Riesz space.

Lemma 29.2. (i) *Let $\{f_\tau\}, \{g_\sigma\}$ be subsets of L such that $\sup_{\tau,\sigma}(f_\tau + g_\sigma) = h$ exists. Then $\sup g_\sigma$ exists if and only if $\sup f_\tau$ exists.*

(ii) *Let $0 \leqslant u_\tau \uparrow u$ and $0 \leqslant v_\sigma \uparrow u$. Then $w_{\tau,\sigma} = \inf(u_\tau, v_\sigma)$ satisfies $0 \leqslant w_{\tau,\sigma} \uparrow u$.*

(iii) *If $A_1, \ldots, A_n$ are ideals in L, then $\{\bigcap_1^n A_k\} = \bigcap_1^n \{A_k\}$.*

(iv) *Let A be an ideal in L. Then $A^p = \{0\}$ if and only if for every $0 \neq f \in L$ there exists $0 \neq g \in A$ such that $|g| \leqslant |f|$.*

Proof. (i) Assume that $\sup f_\tau = f$ exists. Since $f_\tau + g_\sigma \leqslant h$ for all τ, σ, we have $f + g_\sigma \leqslant h$ for all σ, and it follows easily that $\sup_\sigma (f + g_\sigma) = h$. But then $\sup g_\sigma = h - f$.

(ii) Evident.

(iii) By induction it is sufficient to prove that $\{A_1 \cap A_2\} = \{A_1\} \cap \{A_2\}$. Evidently, $\{A_1 \cap A_2\} \subset \{A_1\} \cap \{A_2\}$. For the converse, let $0 \leqslant u \in \{A_1\} \cap \{A_2\}$. Then there exist directed sets $\{u_\tau\} \subset A_1$ and $\{v_\sigma\} \subset A_2$ such that $0 \leqslant u_\tau \uparrow u$ and $0 \leqslant v_\sigma \uparrow u$, hence $w_{\tau,\sigma} = \inf (u_\tau, v_\sigma)$ satisfies $w_{\tau,\sigma} \in A_1 \cap A_2$ and $0 \leqslant w_{\tau,\sigma} \uparrow u$ by (ii). It follows that $u \in \{A_1 \cap A_2\}$.

(iv) Let $A^p = \{0\}$ and $0 \neq f \in L$. If there would not exist any $0 \neq g \in A$ satisfying $|g| \leqslant |f|$, then $\inf (|h|, |f|) = 0$ for every $h \in A$, and so $f \in A^p$. Contradiction. Conversely, let there exist for every $0 \neq f \in L$ an element $0 \neq g \in A$ satisfying $|g| \leqslant |f|$. If now $A^p \neq \{0\}$, there would exist $0 \neq f \in A^p \subset L$, and hence there would also exist $0 \neq g \in A$ such that $|g| \leqslant |f|$. But then $g \in A^p$ in view of $f \in A^p$, and so $g \in A \cap A^p$, which implies $g = 0$. Contradiction.

Definition 29.3. *Let A and B be ideals in the Riesz space L. Then A is said to be order dense in B if $\{A\} \supset B$. In particular, A is said to be order dense in L (or simply order dense) if $\{A\} = L$.*

Lemma 29.4. (i) *A is order dense in B if and only if $\{A\} \supset \{B\}$, i.e., if and only if A is order dense in $\{B\}$. Alternatively, A is order dense in B if and only if $\{A\}$ is order dense in B. In particular, A is order dense if and only if $\{A\}$ is order dense.*

(ii) *If $A \subset B$ and A is order dense in B, then $\{A\} = \{B\}$.*

(iii) *If A is order dense in B and B is order dense in C, then A is order dense in C.*

(iv) *If $A_1, \ldots, A_n$ are order dense, then $\bigcap_1^n A_k$ is order dense.*

(v) *If A is order dense, then $A^p = \{0\}$.*

Proof. The statements in (i) are evident if one observes that $\{\{A\}\} = \{A\}$; (ii) and (iii) follow then immediately from (i). The statement in (iv) follows from $\{\bigcap_1^n A_k\} = \bigcap_1^n \{A_k\}$ which was proved in Lemma 29.2 (iii), and (v) holds in view of $A^p = \{A\}^p = L^p = \{0\}$.

The next theorem characterizes an important class of Riesz spaces.

Theorem 29.5. *In any Riesz space L the following conditions are mutually equivalent.*

(i) *In order that an ideal $A \subset L$ be order dense it is necessary and sufficient that for every $0 \neq f \in L$ there exists $0 \neq g \in A$ satisfying $|g| \leqslant |f|$.*

(ii) *In order that an ideal $A \subset L$ be order dense it is necessary and sufficient that $A^p = \{0\}$.*

(iii) *Every ideal $A \subset L$ satisfies $\{A\} = A^{pp}$.*

(iv) *Every normal subspace $A \subset L$ satisfies $A = A^{pp}$.*

(v) *For every ideal $A \subset L$ we have that $A \oplus A^p$ is order dense.*

(vi) *For every ideal $A \subset L$ we have that $\{A\} \oplus A^p$ is order dense.*

Proof. (i) $\Leftrightarrow$ (ii). This follows from Lemma 29.2 (iv).

(ii) $\Rightarrow$ (iii). Let A be an ideal in L, and let $B = A \oplus A^p$. Then $B^p = A^p \cap A^{pp} = \{0\}$, so $\{B\} = L$ by (ii). Hence, if $0 \leqslant u \in L$, then

$$u = \sup (v + w : 0 \leqslant v \in A,\ 0 \leqslant w \in A^p,\ v + w \leqslant u).$$

This holds in particular for any $0 \leqslant u \in A^{pp}$. But in this case $0 \leqslant w \in A^p$, $w \leqslant u$ implies $w = 0$, so it follows that $u = \sup (v : 0 \leqslant v \in A, v \leqslant u)$. This shows that $u \in \{A\}$. Hence $A^{pp} \subset \{A\}$, and since $\{A\} \subset A^{pp}$ is always satisfied, we have $\{A\} = A^{pp}$.

(iii) $\Rightarrow$ (iv). Evident.

(iv) $\Rightarrow$ (v). Let A be an ideal in L. Then $(A \oplus A^p)^p = A^p \cap A^{pp} = \{0\}$, so $(A \oplus A^p)^{pp} = L$, and hence surely $\{A \oplus A^p\}^{pp} = L$, which implies by (iv) that $\{A \oplus A^p\} = L$, i.e., $A \oplus A^p$ is order dense.

(v) $\Rightarrow$ (vi). Evident.

(vi) $\Rightarrow$ (ii). We need only prove that if $A^p = \{0\}$, then A is order dense, since the converse is always true by Lemma 29.4 (v). Hence, let $A^p = \{0\}$. Then (vi) implies that $\{A\}$ is order dense, i.e. A is order dense by Lemma 29.4 (i).

The plane, ordered lexicographically, is an example of a Riesz space in which none of the conditions of Theorem 29.5 is satisfied.

We have presented conditions under which $\{A\} \oplus A^p$ is order dense. It is also interesting to ask when $\{A\} \oplus A^p = L$. The following theorem gives an answer.

Theorem 29.6. *If A is a given ideal in L, then $L = \{A\} \oplus A^p$ holds if and only if for every $0 \leqslant u \in L$ the corresponding element*

$$u_1 = \sup (v : v \in A,\ 0 \leqslant v \leqslant u) = \sup (v : v \in \{A\},\ 0 \leqslant v \leqslant u)$$

exists.

Proof. It follows immediately from Lemma 29.1 that if one of the supremum expressions exists, then so does the other, and they have the same value. Assume first that, for every $0 \leqslant u \in L$, the corresponding u_1 exists. By the definition of a normal subspace we have $u_1 \in \{A\}$, and hence, in order to show that $L = \{A\} \oplus A^p$, it will be sufficient to prove that $u - u_1 \in A^p$. To this end, we have to show that if $0 \leqslant v \leqslant u - u_1$ and

$v \in A$, then $v=0$. This is easy, since such an element v satisfies $u_1+v \leqslant u$ and $u_1+v \in \{A\}$, so $u_1+v \leqslant u_1$ by the definition of u_1, and hence $v=0$.

Conversely, assume that $L=\{A\} \oplus A^p$, and let $0 \leqslant u \in L$ be given. Then $0 \leqslant u=f_1+f_2$ uniquely with $f_1 \in \{A\}$ and $f_2 \in A^p$. It follows from $-f_1 \leqslant f_2$ that $0 \leqslant f_1^-=(-f_1)^+ \leqslant f_2^+ \in A^p$, so $f_1^- \in A^p$. But also $f_1^- \in \{A\}$, hence $f_1^-=0$, and so $f_1 \geqslant 0$. Similarly $f_2 \geqslant 0$. Thus $0 \leqslant u=u_1+u_2$ uniquely with $0 \leqslant u_1 \in \{A\}$ and $0 \leqslant u_2 \in A^p$. Let $A_1=(v : v \in \{A\},\ 0 \leqslant v \leqslant u)$. For any $v \in A_1$ the decomposition into an element of $\{A\}$ and an element of A^p is obviously $v=v+0$. Hence, $u-v$ has the decomposition $(u_1-v)+u_2$. Since $u-v \geqslant 0$, this implies $u_1-v \geqslant 0$ by the above remarks, i.e., u_1 is an upper bound of A_1. On the other hand $u_1 \in A_1$, so $u_1=\sup A_1$. This is the desired result.

Corollary 29.7. *If $0 \leqslant u \in L$ is given, and A_u is the ideal generated by u, then $L=\{A_u\} \oplus A_u{}^p$ holds if and only if $\sup_n\{\inf(v,nu) : n=1,2,\ldots\}$ exists for all $0 \leqslant v \in L$. Note that if L is σ-Dedekind complete, then $\sup_n \{\inf(v, nu)\}$ exists for every pair $0 \leqslant u,\ v \in L$.*

Proof. Note that $A_u=(f : |f| \leqslant nu$ for some $n=1, 2, \ldots)$. Now, let $0 \leqslant v \in L$, and set $P_v=(w : w \in A_u,\ 0 \leqslant w \leqslant v)$. It is easy to see that

$$P_v=(w : 0 \leqslant w \leqslant \inf (v, nu) \text{ for some } n=1, 2, \ldots).$$

By the preceding theorem we have $L=\{A_u\} \oplus A_u{}^p$ if and only if $\sup P_v$ exists for every v, i.e., if and only if $\sup_n \{\inf (v, nu)\}$ exists for every v.

Definition 29.8. *The Riesz space L is said to be Archimedean if $0 \leqslant u,\ v \in L,\ 0 \leqslant nu \leqslant v$ for $n=1, 2, \ldots$ implies that $u=0$.*

The plane, ordered lexicographically, is not Archimedean (take $u=(0, 1)$ and $v=(1, 0)$).

Lemma 29.9. (i) *L is Archimedean if and only if $\inf_n n^{-1}u=0$ for every $0 \leqslant u \in L$.*

(ii) *L is Archimedean if and only if $f, g \in L,\ nf \leqslant g$ for $n=1, 2, \ldots$ implies that $f \leqslant 0$.*

(iii) *Any Riesz subspace of an Archimedean Riesz space is Archimedean.*

(iv) *If $\sup_n \{\inf (v, nu) : n=1, 2, \ldots\}$ exists for every pair $0 \leqslant u,\ v \in L$, then L is Archimedean. In particular, if L is σ-Dedekind complete, then L is Archimedean.*

(v) *If ${}^0(L^\sim)=\{0\}$, then L is Archimedean. Hence, in particular, every Riesz space endowed with a Riesz norm is Archimedean.*

Proof. (i) Let L be Archimedean, and $0 \leqslant u \in L$. Then $0 \leqslant u_0 \leqslant n^{-1}u$ for $n=1, 2, \ldots$ implies $nu_0 \leqslant u$ for all n, and so $u_0=0$. This shows that $\inf_n n^{-1}u=0$. Conversely, let $\inf_n n^{-1}u=0$ for every $0 \leqslant u \in L$. If now $0 \leqslant nv \leqslant u$ for $n=1, 2, \ldots$, then $v \leqslant n^{-1}u$ for all n, and so $v \leqslant \inf_n n^{-1}u=0$, i.e., $v=0$. This shows that L is Archimedean.

(ii) If the condition is satisfied, then L is evidently Archimedean. Conversely, if L is Archimedean and $nf \leqslant g$ for $n=1, 2, \ldots$, then $0 \leqslant nf^+ \leqslant g^+$ for all n, so $f^+=0$, i.e., $f \leqslant 0$.

(iii) Evident.

(iv) Let $0 \leqslant u, v \in L$ and $nu \leqslant v$ for $n=1, 2, \ldots$. Then $\sup_n \{\inf(nu, v)\} = \sup_n nu$. Since $\sup_n nu = \sup_n 2nu$, it follows that $nu=0$ for all n, and so $u=0$. This shows that L is Archimedean.

(v) Let $0 \leqslant u, v \in L$ and $nu \leqslant v$ for $n=1, 2, \ldots$. For any $0 \leqslant \varphi \in L^\sim$ we have $0 \leqslant \varphi(u) \leqslant n^{-1}\varphi(v)$ for $n=1, 2, \ldots$, so $\varphi(u)=0$. It follows therefore from ${}^0(L^\sim)=\{0\}$ that $u=0$, and this shows that L is Archimedean.

One of the main results about Archimedean Riesz spaces is contained in the following theorem. One half of the theorem, stating that $\{A\}=A^{pp}$ for any ideal A if L is Archimedean, occurs in T. Ogasawara ([1], Ch. 2, § 3, Theorem 2). We are indebted to Professor T. Andô for this reference.

Theorem 29.10. *The Riesz space L is Archimedean if and only if $\{A\}=A^{pp}$ for every ideal $A \subset L$, i.e., if and only if any of the conditions in Theorem* 29.5 *holds.*

Proof. Let L be Archimedean, and A an ideal in L such that $A^p=\{0\}$. By Theorem 29.5 it is sufficient to show that $\{A\}=L$, i.e., that $u=\sup(v : v \in A, v \leqslant u)$ for every $0 \leqslant u \in L$. Set $M_u=(v : v \in A, 0 \leqslant v \leqslant u)$. Obviously, u is an upper bound of M_u; assume the existence of another upper bound w such that $w<u$. From $u-w>0$ and $A^p=\{0\}$ it follows by Lemma 29.2 (iv) that there exists $z \in A$ such that $0<z \leqslant u-w \leqslant u$. Then $z \in M_u$, and so $z \leqslant w$. From $z \leqslant u-w$ and $z \leqslant w$ we derive $2z \leqslant u$. But then $2z \in M_u$, hence $2z \leqslant w$. From $z \leqslant u-w$ and $2z \leqslant w$ it follows that $3z \leqslant u$, so $3z \in M_u$. Proceeding by induction, we obtain $0<nz \leqslant u$ for $n=1, 2, \ldots$, so $z=0$ since L is Archimedean. Contradiction.

For the converse, assume that L satisfies all condition of Theorem 29.5, but L is not Archimedean. Then there exist $u, u_0 \in L$ such that $0<nu \leqslant u_0$ for $n=1, 2, \ldots$. Denoting by A the set of all $f \in L$ satisfying $n|f| \leqslant u_0$ for $n=1, 2, \ldots$, we have $A \neq \{0\}$ since $u \in A$, and it is easy to verify that A is an ideal. In addition, $A=\{A\}$, i.e., A is even a normal subspace. Indeed, let $0 \leqslant v \in \{A\}$, so $v=\sup(w : w \in A, 0 \leqslant w \leqslant v)$. Since $w \leqslant n^{-1}u_0$ for $w \in A$ and $n=1, 2, \ldots$, we have $v \leqslant n^{-1}u_0$ for all n, so $v \in A$. This shows that $A=\{A\}$. Finally, if $A^+=(f : f \in A, f \geqslant 0)$, then $\sup A^+$ does not exist. Indeed, assuming that $\sup A^+=w$, we should have $nv \leqslant w$ for every $v \in A^+$, i.e., $v \leqslant n^{-1}w$ for every $v \in A^+$ and $n=1, 2, \ldots$. This would imply $w \leqslant n^{-1}w$ for $n=1, 2, \ldots$, so $w=0$, contradicting the fact that $A \neq \{0\}$.

By assumption we have $A=\{A\}=A^{pp}$, and $A \oplus A^p$ is order dense. Thus, if $A_0=(v : v \in A^p, 0 \leqslant v \leqslant u_0)$, then $u_0=\sup(A^+ \oplus A_0)$. Since $\sup A^+$ does not exist, it follows from Lemma 29.2 (i) that $\sup A_0$ does not exist. Hence, there exists $w \in L$ such that $0 \leqslant w<u_0$ and $A_0 \leqslant w$. Then u_0-w

is not contained in $A^{pp}=A$. Indeed, if $u_0-w \in A$, then $u_0-w \leqslant \frac{1}{2}u_0$, and so $w \geqslant \frac{1}{2}u_0 \geqslant A^+$. This would imply that $A^+ \oplus A_0 \leqslant w < u_0$, contradicting the fact that $u_0 = \sup (A^+ \oplus A_0)$. It follows now from $u_0 - w$ not being contained in A^{pp} that for some $z \in A^p$ we have $0 < z \leqslant u_0 - w \leqslant u_0$. Hence $z \in A_0$, and so $z \leqslant w$. But then $z \leqslant u_0 - w$ and $z \leqslant w$, so $2z \leqslant u_0$, which implies $2z \in A_0$. Proceeding by induction we obtain $nz \leqslant u_0$ for $n=1, 2, \ldots$, so $z \in A = A^{pp}$. Also $z \in A^p$, so $z=0$. This contradicts $z>0$. The final result is, therefore, that L is Archimedean.

Example 29.11. Let us consider the following list of properties each of which implies the next one.

(i) L is super Dedekind complete.

(ii) L is Dedekind complete.

(iii) L is σ-Dedekind complete.

(iv) For all $0 \leqslant u, v \in L$, the element $\sup_n \{\inf (v, nu)\}$ exists in L.

(v) L is Archimedean.

We will show that each of these properties is properly weaker than the preceding one.

(ii) but not (i). If X is an uncountable point set, and L is the Riesz space of all real bounded functions on X (with the pointwise ordering), then L is Dedekind complete but not super Dedekind complete.

(iii) but not (ii). Let $X=[0, 1]$ and L the Riesz space of all real bounded functions $f(x)$ on X such that $f(x) \neq f(0)$ for at most countably many x. The ordering is the pointwise ordering. Then L is σ-Dedekind complete since $0 \leqslant u_n \uparrow \leqslant u$ implies that $\sup u_n$ is the ordinary pointwise limit of the sequence $u_n(x)$. However, the space L is not Dedekind complete. Indeed, for each $\tau \in [0, 1]$, let $u_\tau(x)$ be equal to τ for $x=\tau$ and equal to zero for $x \neq \tau$. The set of all finite suprema of the functions u_τ is directed upwards and bounded from above, but this set has no least upper bound in L.

(iv) but not (iii). Let $X=\{1, 2, \ldots, \infty\}$, and L the Riesz space of all real bounded functions $f(x)$ on X such that $f(x) \neq f(\infty)$ for at most finitely many x. The ordering is the pointwise ordering. If $0 \leqslant u, v \in L$ are given, then $w = \sup_n \{\inf (v, nu)\}$ exists in L, namely, $w(x)=v(x)$ at all points where $u(x)>0$ and $w(x)=0$ at all points where $u(x)=0$. However, the space L is not σ-Dedekind complete. Indeed, for each $n=1, 2, \ldots$, let $u_n(x)$ be equal to x^{-1} for $x=1, \ldots, n$ and equal to zero at all other points. Then $0 \leqslant u_n \uparrow \leqslant 1$, but the sequence has no least upper bound in L.

(v) but not (iv). The Riesz space $L=C([0, 1])$ of all real continuous functions on $[0, 1]$ is Archimedean, but if $u(x)$ is a function in L which is positive on $[\frac{1}{4}, \frac{3}{4}]$ and vanishes outside this interval, and if $v(x)=1$ for all x, then $\sup_n \{\inf (v, nu)\}$ does not exist in $L=C([0, 1])$. As follows from

Theorem 26.4 in Note VIII, the space $L_\varrho = C([0, 1])$ satisfies $L_\varrho^* = L_\varrho^\sim$, where ϱ is the uniform norm, and in Example 24.5 (ii) in Note VII it was shown that $L_\varrho^* = L_{\varrho,s}^*$, i.e., there do not exist any integrals on $C([0, 1])$. For later purposes it is of some importance to have an example of an Archimedean space L for which condition (iv) is not satisfied, and such that on L there exist plenty of normal integrals. The following example satisfies these conditions.

Let X be an uncountable point set and let L be the Riesz space of all real functions $f(x)$ on X for which there exists a finite number $f(\infty)$ such that, given any $\varepsilon > 0$, we have $|f(x) - f(\infty)| \geqslant \varepsilon$ for at most finitely many x. In other words, if X is topologized by the discrete topology, then L consists of all real continuous functions on X tending to a finite limit as x tends to infinity (this last phrase is to be understood in the sense that the one-point compactification of X is introduced). Evidently, every $f \in L$ is bounded, and $f(x) = f(\infty)$ outside an at most countable subset of X. Furthermore, L is Archimedean. Let $E = \{x_1, x_2, \ldots\}$ be a countable subset of X, and set $u(x_n) = n^{-1}$ and $u(x) = 0$ for all other x. Furthermore, set $v(x) = 1$ on X. Evidently, $0 \leqslant u, v \in L$. Assuming that $w = \sup_n \{\inf (v, nu)\}$ exists in L, we have $w(x) = 1$ on E, and so $w(\infty) = 1$. But this is impossible since for any function satisfying $w(x) = 1$ on E and $w(\infty) = 1$ there is a different $0 \leqslant w'(x) \in L$ satisfying the same conditions such that $w' < w$ in L (indeed, if y is one of the points outside E where $w(y) = 1$, set $w'(y) = 0$ and $w'(x) = w(x)$ at all points $x \neq y$). Hence, L does not satisfy the condition (iv). On the other hand, if $x_0 \in X$ is given, then $\varphi(f) = f(x_0)$ is obviously a normal integral on L. Hence, if $L_n^\sim$ is the collection of all normal integrals on L, and ${}^0(L_n^\sim)$ is the inverse Riesz annihilator of $L_n^\sim$, then ${}^0(L_n^\sim) = \{0\}$.

In connection with the proof of Theorem 29.10, we add the following remark. If L is an arbitrary Riesz space, let A be the set of all $f \in L$ with non-Archimedean behaviour, i.e., A is the set of all $f \in L$ for which there exists $g \in L$ such that $n|f| \leqslant |g|$ for $n = 1, 2, \ldots$. Then A is evidently an ideal and $A \subset N_\varphi$ for every $\varphi \in L^\sim$, where N_φ is the null ideal of φ. Furthermore, if ϱ is a Riesz seminorm on L, then $\varrho(f) = 0$ for every $f \in A$.

Another remark is that Lemma 25.6 can be proved now under weaker hypotheses, as follows.

Lemma 29.12. *Let L_ϱ be a normed Riesz space such that ${}^\perp(L_{\varrho,c}^*) = \{0\}$. Then $(L_\varrho^a)^\perp = L_{\varrho,s}^*$ implies that L_ϱ^a is order dense in L_ϱ.*

Proof. Since L_ϱ is Archimedean by Lemma 29.9 (v), all the properties in Lemma 29.5 hold in L_ϱ, and so it is sufficient to show that $(L_\varrho^a)^p = \{0\}$. Assuming that $(L_\varrho^a)^p \neq \{0\}$, we obtain a contradiction exactly as in the proof of Theorem 25.6.

For a converse result, let once more ${}^{\perp}(L_{\varrho,c}^{*})=\{0\}$, and assume in addition that, for every $0 \leqslant u \in L_{\varrho}$, there exists a sequence $0 \leqslant u_n \in L_{\varrho}^{a}$ such that $0 \leqslant u_n \uparrow u$. Then $\{L_{\varrho}^{a}\}=L_{\varrho}$ holds evidently, and in addition $(L_{\varrho}^{a})^{\perp}=L_{\varrho,s}^{*}$. Indeed, by Theorem 24.6 it is sufficient to prove that $(L_{\varrho}^{a})^{*}=L_{\varrho,c}^{*}$ holds, i.e., that any integral vanishing on L_{ϱ}^{a} vanishes identically on L_{ϱ}. This follows immediately from the hypotheses made. Note that the above-mentioned additional condition is surely satisfied if L_{ϱ} is super Dedekind complete and L_{ϱ}^{a} is order dense in L_{ϱ}.

More generally, if the Riesz space L is super Dedekind complete and A is an ideal in L, then A considered as a Riesz space is super Dedekind complete. If, in addition, A is order dense in L, then there exists for every $0 \leqslant u \in L$ a sequence $u_n \in A$ satisfying $0 \leqslant u_n \uparrow u$. The following theorem goes in the converse direction.

Theorem 29.13. *Let L be σ-Dedekind complete, and A an ideal in L such that A is super Dedekind complete in its own right and such that for any $0 \leqslant u \in L$ there exists a sequence $u_n \in A$ $(n=1, 2, \ldots)$ satisfying $0 \leqslant u_n \uparrow u$. Then L is super Dedekind complete.*

Proof. Let $0 \leqslant u_{\tau} \uparrow \leqslant v$ in L. By hypothesis there exists a sequence $v_n \in A$ satisfying $0 \leqslant v_n \uparrow v$, so $w_{\tau,n}=\inf(u_{\tau}, v_n)$ satisfies $w_{\tau,n} \in A$ and $w_{\tau,n} \uparrow_{\tau} \leqslant v_n \in A$. Hence, since A is Dedekind complete, there exists w_n such that $w_{\tau,n} \uparrow_{\tau} w_n \leqslant v_n$, so that it follows from $w_n \leqslant v$ for all n by the σ-Dedekind completeness of L that $u=\sup w_n$ exists in L. It will be proved now that $u=\sup u_{\tau}$. Since $u \geqslant w_{\tau,n}$ for all τ and n, we have for τ fixed that

$$u \geqslant \sup_n w_{\tau,n}=\sup_n \{\inf(u_{\tau}, v_n)\}=\inf(u_{\tau}, \sup v_n)=\inf(u_{\tau}, v)=u_{\tau},$$

and hence $u \geqslant u_{\tau}$ for all τ. If also $z \geqslant u_{\tau}$ for all τ, then $z \geqslant w_{\tau,n}$ for all τ and n, so $z \geqslant \sup_{\tau} w_{\tau,n}=w_n$, which implies that $z \geqslant \sup w_n=u$. Hence $u=\sup u_{\tau}$, and thus it has been proved that L is Dedekind complete.

It remains to prove that L is super Dedekind complete. Since $w_{\tau,n} \uparrow_{\tau} w_n \leqslant v_n \in A$ and A is super Dedekind complete, there is a sequence $\{\tau_k(n): k=1, 2, \ldots\} \subset \{\tau\}$ such that $w_{\tau_k(n),n} \uparrow_k w_n$, so $\sup_{k,n} w_{\tau_k(n),n}=\sup w_n=u$. Hence, in view of $w_{\tau_k(n),n} \leqslant u_{\tau_k(n)}$, we have also $\sup_{k,n} u_{\tau_k(n)}=u$, so the countable subset $\{u_{\tau_k(n)}\}$ of $\{u_{\tau}\}$ has the same least upper bound as the whole set $\{u_{\tau}\}$. This shows that L is super Dedekind complete.

30. *Extension of order bounded linear functionals*

Let L be an arbitrary Riesz space and A an ideal in L. An element $\varphi \in A^{\sim}$ is said to have an extension if there exists $\psi \in L^{\sim}$ such that $\psi=\varphi$ on A, i.e., if φ is the restriction to A of some $\psi \in L^{\sim}$. If $0 \leqslant \varphi \in A^{\sim}$ and φ has an extension ψ, then φ has a positive extension. Indeed, since $\psi=\varphi$ on A and A is an ideal, we have immediately that $\psi^{+}=\varphi$ on A, and so

ψ^+ is an extension of φ. More generally, if ψ is an extension of φ, then ψ^+ is an extension of φ^+. Indeed, for $0 \leqslant u \in A$ we have

$$\psi^+(u) = \sup\,(\psi(v) : v \in L,\ 0 \leqslant v \leqslant u) = \sup\,(\varphi(v) : v \in A,\ 0 \leqslant v \leqslant u) = \varphi^+(u).$$

Hence, $\varphi \in A^\sim$ has an extension if and only if φ^+ and φ^- have extensions.

The set of all $\varphi \in A^\sim$ which can be extended will be denoted by $A_e^\sim$. If we denote for any $\varphi \in L^\sim$ the restriction of φ to A by φ^A, then $A_e^\sim$ is obviously the set $(\varphi^A : \varphi \in L^\sim)$.

It can happen that $A_e^\sim$ is a proper subset of $A^\sim$. By way of example, let L be the sequence space (c_0) of all real null sequences, and let A be the order dense ideal of all sequences with only a finite number of non-zero coordinates. Then any real sequence represents an element of $A^\sim$, and it even represents a normal integral on A, but it does not necessarily represent an element of $L^\sim = A_e^\sim$. Note, however, that $A_e^\sim = l_1$ is order dense in $A^\sim$.

Theorem 30.1. *If A is an ideal in L, then $A_e^\sim$ is an ideal in $A^\sim$. Even in the case that A is order dense in L, the ideal $A_e^\sim$ is not necessarily order dense in $A^\sim$.*

Proof. We shall prove first that if $0 \leqslant \psi \leqslant \varphi$ with $\psi \in A^\sim$ and $\varphi \in A_e^\sim$, then ψ can be extended. Let φ_0 be a positive extension of φ. Then $|\psi(f)| \leqslant \psi(|f|) \leqslant \varphi_0(|f|)$ for all $f \in A$, and hence, by Theorem 19.2 in Note VI, ψ has a positive extension ψ_0 satisfying $|\psi_0(f)| \leqslant \varphi_0(|f|)$ for all $f \in L$. This shows that $\psi \in A_e^\sim$. It follows easily that $A_e^\sim$ is an ideal in $A^\sim$.

If L is the Riesz space of all Lebesgue measurable (and almost everywhere finite) real functions on $[0, 1]$, and A is the space L_1 of all Lebesgue summable functions on the same interval, then A is an order dense ideal in L. Since $L^\sim = \{0\}$, we have $A_e^\sim = \{0\}$, but $A^\sim$ includes (and is, in fact, equal to) the space L_∞. Hence, $A_e^\sim$ is not order dense in $A^\sim$.

Corollary 30.2. *If ϱ is a Riesz seminorm on L, and A_ϱ^* is the set of all ϱ-bounded linear functionals on A, then A_ϱ^* is an ideal in A_e. Obviously, A_ϱ^* is the set of all restrictions of elements of L_ϱ^* to A.*

Proof. Follows again from Theorem 19.2 in Note VI.

Once again, let A be an ideal in L. By Theorem 21.1 in Note VI, the Riesz annihilator A^0 is a normal subspace of $L^\sim$; hence, if $B = (A^0)^p$ is the normal subspace of all elements in $L^\sim$ disjoint to A^0, then $L^\sim = B \oplus A^0$ by Theorem 17.2 in Note VI. Given $0 \leqslant \varphi \in L^\sim$, we have a unique decomposition $\varphi = \varphi_1 + \varphi_2$ with $0 \leqslant \varphi_1 \in B$ and $0 \leqslant \varphi_2 \in A^0$. Evidently $\varphi_2 = 0$ on A, so $\varphi_1 = \varphi$ on A, i.e., φ_1 is an extension of φ^A, where φ^A is the restriction of φ to A. The component $\varphi_2 \in A^0$ is given by

$$\varphi_2 = \sup\,(\psi : 0 \leqslant \psi \leqslant \varphi,\ \psi = 0 \text{ on } A),$$

so the component $\varphi_1 \in B$ satisfies

$$\varphi_1 = \inf (\psi : 0 \leqslant \psi \leqslant \varphi, \ \psi = \varphi \text{ on } A).$$

In order words, φ_1 is a positive extension of φ^A, and among all positive extensions of φ^A the extension φ_1 is the smallest one in the sense of the order. Evidently, it follows from this last fact that if $0 \leqslant \varphi, \psi \in L^\sim$ and the restrictions φ^A, ψ^A satisfy $\varphi^A = \psi^A$, then the minimal extensions φ_1, ψ_1 satisfy $\varphi_1 = \psi_1$. Observe now that for any $\varphi \in L^\sim$ we have $(\varphi^+)^A = (\varphi^A)^+$ immediately from the definitions, and similarly $(\varphi^-)^A = (\varphi^A)^-$. This implies that if $\varphi, \psi \in L^\sim$ are given (φ and ψ not necessarily positive) and the restrictions φ^A, ψ^A satisfy $\varphi^A = \psi^A$, then the components φ_1 and ψ_1 in B satisfy $\varphi_1 = \psi_1$. In other words, the component φ_1 of φ in B is uniquely determined by the restriction φ^A of φ to A. Hence, we have proved now the following theorem.

Theorem 30.3. *If A is an ideal in L, then there exists a linear mapping $\varphi \to \varphi_m$ of $A_e^\sim$ into $L^\sim$ such that for $0 \leqslant \varphi \in A_e^\sim$ the image φ_m is the unique smallest positive linear extension of φ to L. The mapping is an injection (i.e., different φ have different φ_m), and the set of all images φ_m is the normal subspace $(A^0)^p$ of $L^\sim$.*

In the very special case that L is Dedekind complete and A is a normal subspace of L, we have $L = A \oplus A^p$. Every $\varphi \in A^\sim$ can now be extended, and the special extension φ_m is obtained by setting $\varphi_m = 0$ on A^p. Hence, if $0 \leqslant \varphi \in A^\sim$ and $0 \leqslant u \in L$, then $\varphi_m(u) = \varphi(u_1) = \sup (\varphi(v) : v \in A, 0 \leqslant v \leqslant u)$, where u_1 is the component of u in A. Of course, this does not hold in the general case, but the following remains true.

Theorem 30.4. *If A is an ideal in L, if $0 \leqslant \varphi \in A_e^\sim$ and $0 \leqslant u \in L$, then*

$$\varphi_m(u) = \sup (\varphi(v) : v \in A, \ 0 \leqslant v \leqslant u),$$

and $\varphi_m = 0$ on A^p.

Proof. Given $0 \leqslant \varphi \in A_e^\sim$, we set $\psi(u) = \sup (\varphi(v) : v \in A, \ 0 \leqslant v \leqslant u)$ for every $0 \leqslant u \in L$. Then $\psi(u) = \varphi(u)$ for $0 \leqslant u \in A$ and $\psi(u) \leqslant \varphi_m(u)$ for $0 \leqslant u \in L$. Hence, in order to show that $\psi = \varphi_m$, it is sufficient to prove that $\psi(u_1 + u_2) = \psi(u_1) + \psi(u_2)$ for $0 \leqslant u_1, u_2 \in L$. It is a routine proof to show that $\psi(u_1) + \psi(u_2) \leqslant \psi(u_1 + u_2)$. Conversely, given $\varepsilon > 0$, there exists $v \in A$, $0 \leqslant v \leqslant u_1 + u_2$ such that $\psi(u_1 + u_2) < \varphi(v) + \varepsilon$. Then $v = v_1 + v_2$ with $0 \leqslant v_1 \leqslant u_1$ and $0 \leqslant v_2 \leqslant u_2$, so $v_1, v_2 \in A$ since A is an ideal. Hence

$$\psi(u_1 + u_2) < \varphi(v_1) + \varphi(v_2) + \varepsilon < \psi(u_1) + \psi(u_2) + \varepsilon,$$

and the desired result follows.

Since $0 \leqslant u \in A^p$ implies $v = 0$ for every $v \in A$ satisfying $0 \leqslant v \leqslant u$, we obtain immediately that $\varphi_m = 0$ on A^p.

Theorem 30.5. *The linear one-one mapping $\varphi \to \varphi_m$ of $A_e^\sim$ onto the normal subspace $(A^0)^p$ of $L^\sim$ preserves suprema and infima of arbitrary collections. The same holds for the inverse mapping. In particular,* $(\varphi^+)_m = (\varphi_m)^+$, $(\varphi^-)_m = (\varphi_m)^-$ *and* $|\varphi|_m = |\varphi_m|$. *Furthermore,* $\varphi \perp \psi$ *if and only if* $\varphi_m \perp \psi_m$. *It follows that the Riesz space $A_e^\sim$ is perfect.*

Proof. Since the one-one mapping is linear and order preserving, it follows immediately that suprema and infima of arbitrary collections are preserved. In particular, $(\varphi^+)_m = (\sup(\varphi, 0))_m = \sup(\varphi_m, 0) = (\varphi_m)^+$. Similarly, $(\varphi^-)_m = (\varphi_m)^-$. By addition, $|\varphi|_m = |\varphi_m|$. Furthermore, we have $\varphi \perp \psi \Leftrightarrow \inf(|\varphi|, |\psi|) = 0 \Leftrightarrow (\inf(|\varphi|, |\psi|))_m = 0 \Leftrightarrow \inf(|\varphi|_m, |\psi|_m) = 0 \Leftrightarrow \inf(|\varphi_m|, |\psi_m|) = 0 \Leftrightarrow \varphi_m \perp \psi_m$. Finally, since the normal subspace $(A^0)^p$ of $L^\sim$ is perfect (cf. Corollaries 28.5 and 28.6 in Note VIII), and since $A_e^\sim$ and $(A^0)^p$ are isomorphic, the Riesz space $A_e^\sim$ is also perfect.

Let $A_{c,e}^\sim = A_{e,c}^\sim = A_e^\sim \cap A_c^\sim$ be the collection of all integrals on A which can be extended to order bounded linear functionals on L. The collection $A_{c,e}^\sim$ is an ideal in $A_e^\sim$ as well as in $A_c^\sim$, and hence an ideal in $A^\sim$. Similarly, let $A_{n,e}^\sim = A_{e,n}^\sim = A_e^\sim \cap A_n$ be the collection of all normal integrals on A which can be extended. The collection $A_{n,e}^\sim$ is an ideal in $A_e^\sim$ as well as in $A_n^\sim$, and hence an ideal in $A^\sim$.

Theorem 30.6. (i) *The image of $A_{c,e}^\sim$ under the mapping $\varphi \to \varphi_m$ is the normal subspace $(A^0)^p \cap L_c^\sim$ of $L^\sim$. Hence, $A_{c,e}^\sim$ is perfect.*

(ii) *The image of $A_{n,e}^\sim$ under the mapping $\varphi \to \varphi_m$ is the normal subspace $(A^0)^p \cap L_n^\sim$ of $L^\sim$. Hence, $A_{n,e}^\sim$ is perfect. Furthermore, if ψ is any normal extension to L of $\varphi \in A_{n,e}^\sim$, then $\psi = \varphi_m$ on $\{A\}$, i.e., all normal extensions of φ coincide on $\{A\}$.*

(iii) *If A is order dense in L, then the image of $A_{n,e}^\sim$ is $L_n^\sim$.*

Proof. (i) We need only prove that if $0 \leqslant \varphi \in A_{c,e}^\sim$, then φ_m is an integral on L. To this end, let $0 \leqslant u_n \uparrow u$ in L. Given $\varepsilon > 0$, there exists $v \in A$ such that $0 \leqslant v \leqslant u$ and $\varphi_m(u) < \varphi(v) + \varepsilon$. Since $0 \leqslant v_n = \inf(v, u_n) \leqslant u_n$, we have that $v_n \in A$ for $n = 1, 2, \ldots$, and $0 \leqslant u_n \uparrow u$ implies that $0 \leqslant v_n \uparrow v$, so $\varphi(v_n) \uparrow \varphi(v)$. Hence $\varphi_m(u) < \sup \varphi(v_n) + \varepsilon < \sup \varphi_m(u_n) + \varepsilon$. It follows that $\varphi_m(u_n) \uparrow \varphi_m(u)$, so φ_m is an integral on L.

(ii) The proof that if φ is a normal integral on A which can be extended, then φ_m is a normal integral on L, is similar to the proof in part (i). If $0 \leqslant \varphi \in A_e^\sim$ is normal and ψ is an arbitrary normal extension of φ, then we have for any $0 \leqslant u \in \{A\}$ by Lemma 29.1 that

$$\psi(u) = \sup(\psi(v) : v \in A,\ 0 \leqslant v \leqslant u) = \sup(\varphi(v) : v \in A,\ 0 \leqslant v \leqslant u) = \varphi_m(u).$$

Hence, ψ and φ_m coincide on $\{A\}$.

(iii) If A is order dense in L, i.e., if $\{A\} = L$, then any $\varphi \in A_{n,e}^\sim$ has by part (ii) a unique normal extension to L. Hence, the image of $A^{\sim}_{n.e}$ under the mapping $\varphi \to \varphi_m$ is the whole of $L^{\sim}_n$.

Theorem 30.7. *Let ϱ be a Riesz norm on L_ϱ, and let A be an ideal in L_ϱ. Then the following holds.*

(i) *A^*_ϱ is an ideal in $A^{\sim}_e$, and $\varphi \to \varphi_m$ maps A^*_ϱ onto $(A^0)^p \cap L^*_\varrho = (A^\perp)^p$, where by $(A^\perp)^p$ we denote the normal subspace of L^*_ϱ consisting of all elements of L^*_ϱ disjoint to $A^\perp$.*

(ii) *$\varphi \to \varphi_m$ maps $A^*_{\varrho,c}$ onto the normal subspace $(A^\perp)^p \cap L^*_{\varrho,c}$ of L^*_ϱ.*

(iii) *$\varphi \to \varphi_m$ maps $A^*_{\varrho,n}$ onto the normal subspace $(A^\perp)^p \cap L^*_{\varrho,n}$ of L^*_ϱ. Here $L^*_{\varrho,n}$ denotes the intersection of $(L_\varrho)^{\sim}_n$ and L^*_ϱ, and similarly for $A^*_{\varrho,n}$. If $\varphi \in A^*_{\varrho,n}$, and ψ is a normal extension of φ such that $\psi = 0$ on A^p, then $\psi = \varphi_m$.*

Proof. If $\varphi \in A^*_\varrho$, then φ can obviously be extended to a ϱ-bounded linear functional on L. Hence, if $0 \leqslant \varphi \in A^*_\varrho$, the minimal extension φ_m is surely ϱ-bounded. The remaining statements follow easily, except the last one. For the last statement, let $\varphi \in A^*_{\varrho,n}$, and let ψ be a normal extension of φ such that $\psi = 0$ on A^p. Then $\varphi^+ \in A^*_{\varrho,n}$, and ψ^+ is a normal extension of φ^+ such that $\psi^+ = 0$ on A^p. By the preceding theorem we have $\psi^+ = (\varphi^+)_m$ on $\{A\}$; furthermore, $\psi^+ = (\varphi^+)_m = 0$ on A^p, so $\psi^+ = (\varphi^+)_m$ on $\{A\} \oplus A^p$. But L_ϱ is Archimedean by Lemma 29.9 (v), so $\{A\} \oplus A^p$ is an order dense ideal in L_ϱ by Theorem 29.10; hence, $\psi^+ = (\varphi^+)_m$ on L_ϱ. Similarly $\psi^- = (\varphi^-)_m$ on L_ϱ, and so $\psi = \varphi_m$ on L_ϱ.

31. *Normal subspaces of $L^{\sim}$ and their carriers in L*

In Note VIII (sections 27 and 28) we have proved several results on normal subspaces B of $L^{\sim}$ and their carriers $C_B = ({}^0B)^p$ in L. It was assumed there that L is Dedekind complete. Some of these results can be generalized to the case that L is Archimedean, and not necessarily Dedekind complete. In the present section we include a brief discussion of these more general results, and we begin by proving the following key lemma.

Lemma 31.1. *Let ϱ be a Riesz norm on the Riesz space L.*

(i) *If $0 \leqslant u_n \leqslant u \in L$ for $n = 1, 2, \ldots$ and $\sum \varrho(u_n) < \infty$, then there exist for every $\varepsilon > 0$ and for every $0 \leqslant \varphi \in L^{\sim}$ sequences $\{v_n\}$, $\{w_n\}$, $\{z_n\}$ of positive elements in L such that* (a) $0 \leqslant v_n \leqslant z_n \downarrow 0$, (b) *$\varphi(w_n) < \varepsilon$ for all n,* (c) *$u_n = v_n + w_n$ for all n.*

(ii) *If $0 \leqslant u_n \in L$ and $\sum \varrho(u_n) < \infty$, then the same conclusion holds for every $\varepsilon > 0$ and every $0 \leqslant \varphi \in L^*_\varrho$.*

Proof. (i) Assume the hypotheses of (i) satisfied, and let $\varepsilon > 0$ and $0 \leqslant \varphi \in L^{\sim}$ be given. For every n, set $u_{n,m} = \sup(u_n, \ldots, u_m)$ for $m \geqslant n$.

Then $u_n \leqslant u_{n,m} \uparrow_m \leqslant u$, and so $\varphi(u_{n,m})$ is increasing in m and majorized by $\varphi(u)$. It follows that $\lim\,(\varphi(u_{n,m}) : m \to \infty)$ exists as a finite number. Hence, there exists an index sequence m_n, increasing as n increases, such that, for $n=1, 2, \ldots,$

$$\varphi(u_{n,k}) - \varphi(u_{n,m_n}) < \varepsilon/2^n \text{ for all } k \geqslant m_n.$$

For brevity, set $x_n = u_{n,m_n}$ and $z_n = \inf\,(x_1, \ldots, x_n)$. Then $z_n \downarrow$, and it follows from $z_n \leqslant x_n = u_{n,m_n} \leqslant u_n + \ldots + u_{m_n}$ that $\varrho(z_n) \to 0$ as $n \to \infty$. This shows that $z_n \downarrow 0$.

Now, for every n, let

$$y_n = \sum_{i=1}^{n-1} [\sup\,(x_i, \ldots, x_n) - x_i] = \sum_{i=1}^{n-1} (u_{i,m_n} - u_{i,m_i}).$$

Then $\varphi(y_n) < \sum_{i=1}^{\infty} \varepsilon/2^i = \varepsilon$ for all n. Furthermore, $x_n - y_n \leqslant x_i$ for $i=1, \ldots, n$, and so $x_n - y_n \leqslant z_n$ for $n=1, 2, \ldots$. Since $u_n \leqslant x_n$, this implies that $u_n - y_n \leqslant z_n$, so $u_n \leqslant y_n + z_n$ for all n. Finally, let $v_n = \inf\,(u_n, z_n)$ and $w_n = u_n - v_n$ for all n. Then $0 \leqslant v_n \leqslant z_n \downarrow 0$, and $u_n = v_n + w_n$, so the conditions (a) and (c) are satisfied. For (b), observe that $w_n = u_n - v_n = \inf\,(y_n + z_n, u_n) - \inf\,(z_n, u_n) \leqslant y_n + z_n - z_n = y_n$, so $\varphi(w_n) \leqslant \varphi(y_n) < \varepsilon$ for all n.

(ii) Assume $0 \leqslant u_n \in L$ and $\sum \varrho(u_n) < \infty$, and let $\varepsilon > 0$ and $0 \leqslant \varphi \in L_\varrho^*$ be given. The ϱ^*-bound of φ will be denoted by $\varrho^*(\varphi)$. We have $\varphi(u_{n,m}) \leqslant \sum_n^\infty \varphi(u_i) \leqslant \varrho^*(\varphi) \sum_n^\infty \varrho(u_i) \to 0$ as $n \to \infty$, so $\lim\,(\varphi(u_{n,m}) : m \to \infty)$ exists as a finite number. The remaining part of the proof is the same as in part (i). A second proof is as follows. There is an index N_ε such that $\varphi(u_n) < \varepsilon$ for $n \geqslant N$. For $n > N$, take $v_n = z_n = 0$, $w_n = u_n$, and for $n = 1, \ldots, N$ take $w_n = 0$, $v_n = u_n$ and $z_n = \sup\,(u_n, u_{n+1}, \ldots, u_N)$.

Let L be an arbitrary Riesz space, and let $\varphi \in L^\sim$. We recall that the null ideal N_φ of φ is the ideal in L consisting of all $f \in L$ satisfying $|\varphi|(|f|) = 0$, and the carrier C_φ of φ is the normal subspace N_φ^p of L. Evidently $N_\varphi = N_{|\varphi|}$, and so $C_\varphi = C_{|\varphi|}$; furthermore, if φ is a normal integral, then N_φ is a normal subspace. Given $\varphi, \psi \in L^\sim$, it is obvious that $C_\varphi \subset N_\psi$ implies $C_\varphi \perp C_\psi$ and $C_\varphi \subset (N_\psi)^{pp}$. In fact, the following three conditions are always mutually equivalent: (i) $C_\varphi \subset (N_\psi)^{pp}$, (ii) $C_\varphi \perp C_\psi$, (iii) $C_\psi \subset (N_\varphi)^{pp}$. The following example shows, however, that $C_\psi \subset N_\varphi$ does not always imply $C_\varphi \subset N_\psi$. Let $L = C([0, 1])$, φ the Lebesgue integral and ψ the point functional defined by $\psi(f) = f(\frac{1}{2})$. Then $N_\varphi = \{0\}$, $C_\varphi = L$, $N_\psi = (f : f(\frac{1}{2}) = 0)$ and $C_\psi = \{0\}$. Hence $C_\psi \subset N_\varphi$, but $C_\varphi \subset N_\psi$ does not hold.

In Theorem 27.6 (Note VIII) it was shown that if L is Dedekind complete and $\varphi, \psi \in L_n^\sim$, then the equivalent conditions (i), (ii), (iii) above are also equivalent to $\varphi \perp \psi$. This result can be improved in the following way.

Theorem 31.2. (i) *Let L be an arbitrary Riesz space. If $\varphi \perp \psi$ with $\varphi \in L^\sim$ and $\psi \in L_c^\sim$, then $C_\varphi \subset N_\psi$ (and hence $C_\varphi \perp C_\psi$ and $C_\psi \subset (N_\varphi)^{pp}$).*

(ii) *If L is Archimedean and if $\varphi, \psi \in L_n^\sim$, then the following conditions are mutually equivalent*: (a) $\varphi \perp \psi$, (b) $C_\varphi \subset N_\psi$, (c) $C_\psi \subset N_\varphi$, (d) $C_\varphi \perp C_\psi$.

Proof. (i) We may assume that $\varphi, \psi \geqslant 0$. Given $0 \leqslant u \in C_\varphi$, it follows from $\{\inf(\varphi, \psi)\}(u) = 0$ that there exist sequences $\{u_n\}$, $\{u_n'\}$ of positive elements such that $u = u_n + u_n'$ and $\varphi(u_n) + \psi(u_n') \leqslant 2^{-n}$ for $n = 1, 2, \ldots$. Hence $\sum \varphi(u_n) < \infty$, and $0 \leqslant u_n \leqslant u$ for all n. Since $\varrho(f) = \varphi(|f|)$ is a norm on C_φ it follows now from part (i) of the preceding lemma that for every $\varepsilon > 0$ there are sequences $\{z_n\}$, $\{v_n\}$, $\{w_n\}$ such that $0 \leqslant v_n \leqslant z_n \downarrow 0$, $\psi(w_n) < \varepsilon$ for all n, and $u_n = v_n + w_n$ for all n. Then $u_n' = u - u_n = u - v_n - w_n \geqslant 0$ implies that $\psi(u - v_n) = \psi(u_n' + w_n) < 2^{-n} + \varepsilon$ for all n, and $0 \leqslant v_n \leqslant z_n \downarrow 0$ implies that $u \geqslant u - v_n \geqslant u - z_n \uparrow u$. Hence, since $\psi \in L_c^\sim$, we have $\psi(u - v_n) \rightarrow \psi(u)$ as $n \rightarrow \infty$, so $\psi(u) \leqslant \varepsilon$. This holds for every $\varepsilon > 0$, so $\psi(u) = 0$. It has been shown thus that $C_\varphi \subset N_\psi$.

(ii) Since L is now Archimedean, we have $\{A\} = A^{pp}$ for every ideal A; in particular, $N_\varphi = \{N_\varphi\} = (N_\varphi)^{pp}$ for every $\varphi \in L_n^\sim$. Thus (b), (c), (d) are mutually equivalent, and by part (i) are implied by (a). For the converse, assume $0 \leqslant \varphi, \psi \in L_n^\sim$ and (b), (c), (d) satisfied. Then $0 \leqslant \varphi_0 = \inf(\varphi, \psi) \in L_n^\sim$ and $\varphi_0 = 0$ on the ideal $C_\varphi \oplus N_\varphi$. Since L is Archimedean, the ideal $C_\varphi \oplus N_\varphi$ is order dense in L, and so $\varphi_0 = 0$ on L. This completes the proof.

Corollary 31.3 (*Hahn decomposition*). *Let L be Archimedean. Given $\varphi \in L_n^\sim$, there exist normal subspaces P and N such that $P \cap N = \{0\}$, $P \oplus N$ is order dense in L, $\varphi(u) \geqslant 0$ for all $0 \leqslant u \in P$ and $\varphi(u) \leqslant 0$ for all $0 \leqslant u \in N$. More precisely, we can choose $P = C_{\varphi+}$ and $N = \{N_{\varphi+}\} = N_{\varphi+}$. For $\varphi \in L_c^\sim$ the same holds except that in this case we can only assert that P and N are ideals one at least of which is a normal subspace.*

Proof. Given $\varphi \in L_c^\sim$, the fact that $\varphi^+ \perp \varphi^-$ implies by part (i) of the theorem that $\varphi^-(C_{\varphi+}) = 0$ and $\varphi^+(C_{\varphi-}) = 0$. Hence $P = C_{\varphi+}$ is a normal subspace, $N = N_{\varphi+}$ is an ideal, and $P \oplus N$ is order dense in L, so P and N satisfy the desired conditions.

Lemma 27.8 of Note VIII can be generalized as follows.

Lemma 31.4. *Let L be Archimedean. If $0 \leqslant \varphi \in L_n^\sim$ and $0 \leqslant u \in L$ are such that $\varphi(u) > 0$, then there exists for every $\varepsilon > 0$ an element $0 \leqslant v_\varepsilon \leqslant u$ such that $\varphi(u) - \varepsilon \leqslant \varphi(v_\varepsilon) \leqslant \varphi(u)$ and $\psi(v_\varepsilon) = 0$ for all $\psi \in L_c^\sim$ such that $\psi \perp \varphi$.*

Proof. Since $C_\varphi \oplus N_\varphi$ is order dense, we have

$$u = \sup(v + w : 0 \leqslant v \in C_\varphi,\ 0 \leqslant w \in N_\varphi,\ v + w \leqslant u),$$

and hence $\varphi(u) = \sup(\varphi(v) : 0 \leqslant v \in C_\varphi,\ v \leqslant u)$. Given $\varepsilon > 0$, it follows that $\varphi(u) - \varepsilon \leqslant \varphi(v_\varepsilon) \leqslant \varphi(u)$ for some $0 < v_\varepsilon \in C_\varphi$, $v_\varepsilon \leqslant u$. In addition, given $\psi \in L_c^\sim$ such that $\psi \perp \varphi$, it follows from part (i) of the preceding theorem that $C_\varphi \subset N_\psi$, so $\psi(v_\varepsilon) = 0$.

Theorem 27.10 and Corollary 27.11 can also be generalized. We recall that $A^{\#}=A^0 \cap L_n^\sim$ for any arbitrary subset A of L.

Theorem 31.5. *Let L be Archimedean. If B is an ideal in $L_n^\sim$, then $({}^0B)^{\#}=\{B\}$, where by $\{B\}$ we mean the normal subspace of $L_n^\sim$ generated by B.*

Proof. Evidently $({}^0B)^{\#}$ is a normal subspace of $L_n^\sim$, and $\{B\} \subset ({}^0B)^{\#}=$ $=({}^0\{B\})^{\#}$ on account of ${}^0B={}^0\{B\}$. Hence, it is sufficient to consider the case that B is a normal subspace of $L_n^\sim$. The proof is then the same as for Theorem 27.10.

Corollary 31.6. (i) *If L is Archimedean, and B_1 and B_2 are ideals in $L_n^\sim$, then $\{B_1\} \subset \{B_2\}$ if and only if ${}^0B_1 \supset {}^0B_2$. In particular, $\{B_1\}=\{B_2\}$ if and only if ${}^0B_1={}^0B_2$.*

(ii) (*Weak Radon–Nikodym theorem*). *If L is Archimedean, $\varphi, \psi \in L_n^\sim$, and Φ, Ψ are the normal subspaces of $L_n^\sim$ generated by φ and ψ respectively, then $\psi \in \Phi$ if and only if ${}^0\Phi \subset {}^0\Psi$.*

The next theorem was not yet mentioned in Note VIII.

Theorem 31.7. (i) *Let A be an ideal in the Archimedean Riesz space L. Then $A_{n,e}^\sim$ is order dense in $A_n^\sim$ if and only if ${}^0(A_{n,e}^\sim)={}^0(A_n^\sim)$.*

(ii) *Let A be an ideal in the Riesz space L. If ${}^0(A_{n,e}^\sim)=\{0\}$, then $A_{n,e}^\sim$ is order dense in $A_n^\sim$. In particular, if ${}^0(L_n^\sim)=\{0\}$, then $A_{n,e}^\sim$ is order dense in $A_n^\sim$ for every ideal $A \subset L$.*

Proof. (i) Follows immediately from Corollary 31.6 (i).

(ii) If ${}^0(A_{n,e}^\sim)=\{0\}$, then ${}^0(A_n^\sim)=\{0\}$, and so surely ${}^0(A^\sim)=\{0\}$. It follows from Lemma 29.9 (v) that A is Archimedean, and so, since ${}^0(A_{n,e}^\sim)={}^0(A_n^\sim)$, we have that $A_{n,e}^\sim$ is order dense in $A_n^\sim$. If ${}^0(L_n^\sim)=\{0\}$, then L is Archimedean, and so every ideal $A \subset L$ is Archimedean. Hence, since ${}^0(L_n^\sim)=\{0\}$ implies also that ${}^0(A_{n,e}^\sim)={}^0(A_n^\sim)=\{0\}$, the desired result follows.

If B is an ideal in $L_n^\sim$, then the normal subspace of L generated by all carriers C_φ, $\varphi \in B$, is called the carrier of B, and denoted by C_B. The following theorem generalizes Theorem 27.12 in Note VIII.

Theorem 31.8. (i) *If L is Archimedean and B is an ideal in $L_n^\sim$, then $C_B=({}^0B)^p$. Hence $L=\{{}^0B \oplus C_B\}$. In addition, the ideals B and B_1 in $L_n^\sim$ are disjoint if and only if C_B and C_{B_1} are disjoint.*

(ii) *If L is Archimedean and B is an ideal in L_n, then $C_B=C_{\{B\}}$.*

Proof. (i) The proof is almost the same as for Theorem 27.12.

(ii) Since ${}^0B={}^0\{B\}$, we have $C_B=({}^0B)^p=({}^0\{B\})^p=C_{\{B\}}$.

Still assuming that L is Archimedean, let $L_0={}^0(L_n^\sim)$ and let L_1 be the carrier of $L_n^\sim$, so $L_1=(L_0)^p$ and $L=\{L_0 \oplus L_1\}$. Evidently, the carrier of any ideal $B \subset L_n^\sim$ is included in L_1, and the space $(L_1)_n^\sim$ of all normal integrals on L_1 may be identified in an obvious manner with $L_n^\sim$, since L is Archimedean. Lemma 31.9 and Theorem 31.10 which follow are generalizations of Lemma 27.13 and Theorem 27.14 respectively.

Lemma 31.9. *If L is Archimedean, A a normal subspace of L_1, and $u \geqslant 0$ is an element of L_1 which is not in A, then there exists $0 \leqslant \varphi \in L_n^\sim$ such that $\varphi = 0$ on A and $\varphi(u) > 0$.*

Proof. Since $\{A \oplus (A^p \cap L_1)\} = L_1$, the given element $u \geqslant 0$ satisfies

$$u = \sup (v+w : 0 \leqslant v \in A,\ 0 \leqslant w \in A^p \cap L_1,\ v+w \leqslant u).$$

Hence, if all $w \in A^p \cap L_1$ such that $0 \leqslant w \leqslant u$ would satisfy $w=0$, then $u \in A$, since A is a normal subspace. This is contrary to hypothesis, so there exists $w \in A^p \cap L_1$ with $0 < w \leqslant u$. But then, on account of ${}^0(L_n^\sim) = L_0$, there exists $0 \leqslant \varphi \in L_n^\sim$ such that $\varphi(w) > 0$, and in view of Lemma 19.5 (Note VI) we may assume that $\varphi(z) = 0$ for all $z \perp w$. It follows that $\varphi = 0$ on A and $\varphi(u) \geqslant \varphi(w) > 0$.

Theorem 31.10. (i) *If L is Archimedean and A is a normal subspace of L_1, then ${}^0(A^\#) = \{A \oplus L_0\}$.*

(ii) *If L is Archimedean and A is a normal subspace of L_1, then A is the carrier of some normal subspace of $L_n^\sim$.*

Proof. (i) Evidently ${}^0(A^\#) \supset \{A \oplus L_0\}$. Conversely, if $u > 0$ is not in $\{A \oplus L_0\}$, then it follows similarly as in the preceding lemma that there exists $0 < v \leqslant u$ such that $v \in \{A \oplus L_0\}^p$, so v is an element of L_1 which is not in A. Hence, by the preceding lemma, there exists $0 \leqslant \varphi \in L_n$ such that $\varphi = 0$ on A and $\varphi(v) > 0$, so $\varphi(u) \geqslant \varphi(v) > 0$. This shows that u is not in ${}^0(A^\#)$. The same result follows then easily for any (not necessarily positive) $f \in L$ which is not in $\{A \oplus L_0\}$.

(ii) Given the normal subspace $A \subset L_1$, we have $L_1 = \{A \oplus A_1\}$, where $A_1 = A^p \cap L_1$. From (i) it follows that ${}^0(A_1{}^\#) = \{A_1 \oplus L_0\}$, so $({}^0(A_1{}^\#))^p = \{A_1 \oplus L_0\}^p = (A_1 \oplus L_0)^p = A_1{}^p \cap L_1 = A^{pp} = A$. This shows that A is the carrier of $A_1{}^\#$.

Observe that if L is Archimedean, then by Theorem 31.5 every normal subspace of $L_n^\sim$ is closed in the $\sigma(L_n^\sim, L)$ topology, and by part (ii) of Theorem 31.10 every normal subspace of L_1 is closed in the $\sigma(L_1, L_n^\sim)$ topology. Finally, we note that the formulas of the "carrier calculus", presented in Theorem 27.15, extend to the case that L is Archimedean.

(To be continued)

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